

value is uncertain; Gaydon⁴ has suggested $4.4 + 0.1$ v.e.

The value 6.34 v.e. for $D_{\text{extrap.}}$ for SO leads on correction by $0.37 + 0.66$ for the valence states of the two atoms to $D_0 \cong 5.31$ v.e., in approximate agreement with the precisely known value 5.184 v.e.

The valence state for nitrogen, at $27/100 F^2$ (with 2D at $9/25 F^2$ and 2P at $3/5 F^2$), is calculated to lie about 1.67 v.e. above the normal state, 4S , that for the iso-electronic oxygen ion O^+ is 2.34 v.e., and that for phosphorus is 1.05 v.e. above their normal states. Similarly the bivalent states of carbon, $:\dot{C}\cdot$, the nitrogen ion, $:\dot{N}^+$, and silicon, $:\dot{Si}\cdot$, are 0.44 v.e., 0.64 v.e., and 0.28 v.e., respectively, above the 3P normal states. The valence corrections together with the values of $D_{\text{extrap.}}$ lead to values of the dissociation energy in rough agreement with the directly determined values for many molecules.

It is interesting to consider the nitrogen molecule and the carbon monoxide molecule, the dissociation energies of which remain uncertain despite much work and discussion. Values of D_0 for N_2 between 5.2 and 11.9 v.e. have been suggested in recent years, the most likely⁴ being 7.383, 8.573, and 9.764 v.e. We note that $D_{\text{extrap.}}$ diminished by the valence correction for two nitrogen atoms is 8.46 v.e., supporting the value 8.573 v.e. Similarly the value 9.75 v.e. is found for carbon monoxide, which supports the predissociation value⁵ 9.847 v.e., rather than the alternatives 11.11, 9.14, and 6.92 v.e. that have been proposed. However, the

⁴ A. G. Gaydon, *Dissociation Energies*, Chapman and Hall, Ltd., London 1947.

values of $D_{\text{extrap.}} + A_0$ for the first three excited states indicate 11.1 v.e. for D_0 .

A rough correlation between $D_{\text{extrap.}}$ and bond type is evident for the more stable states of the diatomic molecules. Thus the bonds $A \equiv A$ and $A = A$ between elements of the first short period tend to have dissociation energy to the atomic valence state equal to about 6.6 v.e. Examples are $O_2^+ X$, 6.51; $N_2 B$, 6.68; $N_2 a'$, 6.56; $C_2 A$, 7.05; $C_2 b$, 6.55 v.e. An increase, presumably due to the stabilizing effect of the partial ionic character of the double bond, is observed when the atoms differ by 0.5 in electronegativity: $NO X$, 7.69; $CN A$, 7.62 v.e.

There is striking agreement, at 7.84 ± 0.02 v.e., among the values of $D_{\text{extrap.}}$ for the normal states of SC, PN, and SiO, with the structures $:A \equiv B:$. The agreement extends even to the excited state $A^2\Sigma$ of PO, with a promoted outer electron, $(:P=O:)\cdot$. The presence of two outer σ electron pairs is seen to stabilize the molecule, since $D_{\text{extrap.}}$ is smaller for triple-bonded structures with an odd outer σ electron — 6.90 for PC X, and 6.21 for SiN X. This is surprising; the resonance of the outer odd electron and electron pair, $\{ :A \equiv B: , :A \equiv B\cdot \}$, might be expected to contribute to the stabilization of the normal state, and to make such a bond stronger than the ordinary triple bond.

⁵ H. D. Hagstrum and J. T. Tate, *Physic. Rev.* **59**, 354 [1941], have stated that 9.6 ± 0.2 v.e. is the only value which yields a consistent interpretation of all of the electron-impact observations made on carbon monoxide.

Polyhedral Harmonics

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It is a familiar fact and one which is frequently referred too in books on the methods of theoretical physics that solutions of the wave equation are possible in closed form only for domains of a very limited number of shapes. In the sequel, a solution for a region is meant to be a solution, which either vanishes itself or whose normal

derivative vanishes on the boundary of that region. The limitation just referred to may concern both the shape of the bounding surface and the range of the curvilinear coordinates, which vary along the surface. As an example of the first type of restriction we note that solutions of the wave equations are known only for domains bounded



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by quadric surfaces or by degenerate cases of these. An example of a restriction of the second kind is the fact that solutions of the two dimensional wave equation for flat triangular regions may be written down only for the equilateral case and for a few triangles like $(1/2, 1/4, 1/4)\pi$ and $(1/2, 1/3, 1/6)\pi$.

The property, which distinguishes these triangles from all others, is obviously their special symmetry. This is further brought out by the fact that for circular sectors or for regions bounded by two straight lines the vertex angle has an essential influence upon the character of the solution. As long as the vertex angle is a submultiple of π , i.e. π/n with n integral, the solution is a univalued function, while for a general angle it is of a much more complicated character. Obviously what distinguishes the former regions from the latter is that by repeated reflections the whole plane (or the whole angular region of 2π radians) may be covered.

The following remarks are concerned with the generalization of the last mentioned case into three dimensional space. The problem to be discussed is the following: For which regions bounded by planes that meet in one point is it possible to find simple solutions of the wave equation? Upon introducing spatial polar coordinates, whose origin is at the point of intersection of the planes, it is seen that the problem is equivalent to that of finding solutions of the three dimensional Laplace-equation, which either vanish or whose normal derivatives vanish upon planes meeting in one point.

The only cases of this kind for which up to the present harmonic functions have been found, are solid angular regions formed by the symmetry planes of one of the four groups associated with the regular solids. Just as in the two dimensional case those circular harmonics are known to be of especially simple form, which possess the symmetry of a regular n -gon, so in space we are

led to the spherical harmonics possessing the symmetry of the regular solids. The symmetry planes can also be said to intersect the sphere of radius unity in great circles, which in turn subdivide the surface of the sphere into a certain number of congruent spherical triangles.

The invariants of the regular polyhedra have been discussed since the book by F. Klein¹ on the connection between the Icosahedron and the equation of fifth degree. Surfaces possessing the symmetry of the polyhedra were studied systematically by Goursat². The need for spherical harmonics possessing this symmetry was first pointed out by Pockels³ in his classic book. He gave, without proof, the simplest harmonics, which vanish at the planes of symmetry. Higher harmonics of this kind were not given, nor were the harmonics, whose normal derivative vanishes, discussed. Polyhedral groups and spherical harmonics are studied in detail in two papers by Elert⁴ and Bethe⁵. However, the problem dealt with by these authors — that of the transformations of spherical harmonics into one another by means of the tetrahedral and octahedral groups and the representations of these groups generated thereby — is only remotely connected with the present problem. Two papers, by Poole⁶ and Hodgkinson⁷, constitute the first contribution to the problem since Pockels' early results, in as much as rules are given for the construction of such harmonics, whose normal derivatives vanish on the symmetry planes. These authors were evidently unaware of the results of Goursat and Pockels, for the developments of especially the former paper are — though in different language — repeated. Furthermore no mention is made of a second class of polyhedral harmonics, which for applications is just as important: namely those vanishing on the symmetry planes. There is, finally, a recent paper by Beckenbach and Reade⁸, in which mean value theorems and displacement properties of a certain class of harmonic functions related to polyhedral harmonics are studied. As a last remark it should be pointed out that the harmonics discussed in this paper are quite different from the tetrahedral and octahedral eigenfunctions introduced by Pauling⁹. Paulings functions are sums of spherical surface harmonics for $l = 0, 1$ and 2 and are therefore not spherical harmonics in the proper sense; we, on the other hand, are interested in homogeneous solutions of Laplaces equation for a definite degree l .

¹ F. Klein, Vorlesungen über das Ikosaeder. Leipzig 1884.

² E. Goursat, Étude des surfaces qui admettent tous les plans de symétrie d'un polyèdre régulier. Ann. Ecole Norm. Sup. (3), 4, 158 [1887].

³ F. Pockels, Über die Differentialgleichung $\Delta u + k^2 u = 0$. Leipzig 1891. Esp. pp. 145 and 156—158. It is remarkable that other standard books such as Heine, Thompson and Tait, or Hobson do not mention polyhedral harmonics.

⁴ W. Elert, Z. Physik 51, 6 [1928].

⁵ H. Bethe, Ann. Physik 3, 133 [1929].

⁶ E. G. C. Poole, Spherical harmonics having polyhedral symmetry. Proc. London Math. Soc. (2) 33, 435 [1932].

⁷ J. Hodgkinson, J. London Math. Soc. 10, 221 [1935].

⁸ E. F. Beckenbach and M. Reade, Duke Math. J. 12, 629 [1945].

⁹ L. Pauling, The Nature of the Chemical Bond. Ithaca 1940; esp. Chapter III.

1. Polyhedral groups and their solid angular regions

In the following developments rules are given for the construction of harmonic polynomials, which are invariant under the substitutions of the four different polyhedral groups. The linear partial differential equations of physics call for solutions, for which either the normal derivative or the function itself vanishes on the boundary of the region. It is clear therefore, that in the present case where the boundaries are the symmetry planes of the polyhedra, the former harmonic behaves symmetrically upon passage through such a plane, while the latter type of harmonic changes sign in adjacent regions. The former type will be called the even harmonic, the latter the odd harmonic¹⁰.

Table 1 exhibits the four cases, which may be treated by means of the polyhedral groups.

Group	Angles of spherical triangle	Area	Number of triangles on sphere
Dihedral case	$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{n}\right)\pi$	$\frac{1}{n}\pi$	$4n$
Tetrahedral case	$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}\right)\pi$	$\frac{1}{6}\pi$	24
Octahedral case	$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)\pi$	$\frac{1}{12}\pi$	48
Icosahedral case	$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right)\pi$	$\frac{1}{30}\pi$	120

Table 1.

In the first column the polyhedron in question is written down as a label for each case. It is well known that to each polyhedron a dual one can be constructed possessing as many faces and corners as the original polyhedron possessed corners and faces respectively. One simply constructs planes at the corners, which in each case are perpendicular to the rays from the centre to the corners, and allows those planes to grow until they intersect each with its neighbour. Dual polyhedra possess identical symmetry groups. In this sense cube and octahedron are dual to one

another, also icosahedron and dodecahedron. The second case is self dual, i.e. the dual of a tetrahedron is again a tetrahedron.

Concerning the first case mentioned in the table, a separate word may be necessary. The polyhedron, which goes with this case, is a double pyramid, whose base is a regular polygon. For each value of n , one therefore arrives at a new triangle and the division of the sphere, which results, is like that of a globe, on which the equator and n meridians intersecting under equal angles at the north pole have been drawn.

An important special case of practical interest is obtained for $n=2$; the division of the sphere is into eight rectangular triangles, which are simply the eight octants of a cartesian coordinate system. This group is often called the four-group.

The symmetry planes of the various polyhedra divide the surface of the sphere into congruent areas, which are recorded in the second column. The explanation for the fact that the tetrahedron divides the sphere into 24 triangles of great circles making the angles $\pi/2, \pi/3, \pi/3$ with one another, rather than into 4 equilateral triangles with angles $2\pi/3, 2\pi/3, 2\pi/3$ lies of course in the fact that each symmetry plane, which goes through an edge, will bisect the opposite equilateral triangle. Since three such bisectors go through each face, we get the sphere division shown in the table. Similarly for the other two cases.

The third column contains the area of each spherical triangle according to the formula:

$$A = \alpha + \beta + \gamma - \pi,$$

where α, β, γ are the angles between the symmetry planes or between the great circles. The number of such triangles filling the surface of the sphere, recorded in the last column, is found by dividing the above area into 4π . It is seen that for the proper regular bodies this number can always be found by multiplying the number of triangular faces (tetrahedron: 4, octahedron: 8, icosahedron: 20) by six, for as said earlier, each face is divided into six triangles by its three medians.

2. Polyhedral Invariants

In the following tables the salient facts concerning each of the four groups are set forth. The first entry shows the substitutions, which constitute the group. Next there are recorded the

¹⁰ For electromagnetic fields it can be proved that for arbitrary cone-shaped regions the odd harmonic gives rise to fields, which lack a radial component of H , the even harmonic to fields, which lack the radial E component. See O. Laporte, Amer. J. Physics [1948].

five invariants. By an invariant there is meant a homogeneous polynomial in x, y, z , which is reproduced when any of the linear transformations of the polyhedral group in question is performed. The invariants are denoted by φ or Φ ; and the subscript indicates the degree. Two of the five invariants of each case are trivial, viz. 1 and $x^2 + y^2 + z^2$. The third and fourth are of degree ≥ 2 and are characteristic of the polyhedron in question. It should be noted that the number of substitutions (i.e. the order of the group) is equal to the product of the degrees of the invariants φ . The fifth invariant, denoted by Φ , is of higher degree and is obtained as Jacobian of the three preceding ones. These invariants, which can be derived in straightforward fashion by carrying out the symmetry operations of the group, are given in the works of Klein¹ and Goursat². Following the latter author we also use the abbreviations $s = x + iy$; $s_0 = x - iy$.

The above invariants may be called primitive invariants. Others are obtained as linear combinations of powers and products of these. There is consequently a certain amount of freedom in the choice of the primitive invariants; for instance in the tetrahedral case one might add to φ_4 some constant multiple of φ_2^2 . Nothing essentially new would result.

An important feature is the exceptional position of the invariant Φ . Let us consider the behavior of the invariants as the point xyz passes through any of the symmetry planes¹¹. It is seen that all

120 substitutions¹²:

$$\begin{aligned}\varphi_0 &= 1, & \varphi_2 &= x^2 + y^2 + z^2, \\ \varphi_6 &= z^6 - 5s s_0 z^4 + 5s^2 s_0^2 z^2 - (s^5 + s_0^5)z, \\ \varphi^{10} &= z^{10} - 35s s_0 z^8 + 260s^2 s_0^2 z^6 - 175s^3 s_0^3 z^4 + 25s^4 s_0^4 z^2 \\ &\quad + (s^5 + s_0^5)(123z^5 - 90s s_0 z^3 + 15s^2 s_0^2 z) + (s^5 + s_0^5)^2, \\ \Phi_{15} &= (s^5 - s_0^5) \{ -1024z^{10} + 3840s s_0 z^8 - 3840s^2 s_0^2 z^6 + 1200s^3 s_0^3 z^4 - 100s^4 s_0^4 z^2 \\ &\quad + (s^5 + s_0^5)(-352z^5 - 160s s_0 z^3 - 10s^2 s_0^2 z + s^5 + s_0^5) \} \\ &= \prod_{\nu=0}^4 \left(x \sin \frac{2\pi\nu}{5} + y \cos \frac{2\pi\nu}{5} \right) \left(2z \cos \frac{2\pi}{5} + x \cos \frac{2\pi\nu}{5} - y \sin \frac{2\pi\nu}{5} \right) \\ &\quad \cdot \left(2z \cos \frac{\pi}{5} - x \cos \frac{2\pi\nu}{5} + y \sin \frac{2\pi\nu}{5} \right).\end{aligned}$$

Table 5. Icosahedral case.

4n substitutions

$$\left. \begin{aligned} s, s_0, \pm z; & s e^{2\pi i k/n}, s_0 e^{-2\pi i k/n}, \pm z \\ s_0, s, \pm z; & s_0 e^{-2\pi i k/n}, s e^{2\pi i k/n}, \pm z \end{aligned} \right\} k=1, \dots, n-1$$

$$\begin{aligned}\varphi_0 &= 1, & \varphi_2 &= x^2 + y^2 + z^2, \\ \varphi_2' &= z^2, & \varphi_n &= s^n + s_0^n,\end{aligned}$$

$$\Phi_{n+1} = \frac{1}{2i} z (s^n - s_0^n).$$

Table 2. Dihedral case.

24 substitutions:

$$x; y, z; \quad x, -y, -z; \quad -x, y, -z; \quad -x, -y, z$$

and five permutations of each of these.

$$\begin{aligned}\varphi_0 &= 1, & \varphi_2 &= x^2 + y^2 + z^2, \\ \varphi_3 &= xyz, & \varphi_4 &= x^2 y^2 + y^2 z^2 + z^2 x^2, \\ \Phi_6 &= (x^2 - y^2)(y^2 - z^2)(z^2 - x^2).\end{aligned}$$

Table 3. Tetrahedral case.

48 substitutions:

$$\pm x, \pm y, \pm z$$

all eight combinations of signs and five more permutations.

$$\begin{aligned}\varphi_0 &= 1, & \varphi_2 &= x^2 + y^2 + z^2, \\ \varphi_4 &= x^2 y^2 + y^2 z^2 + z^2 x^2, & \varphi_6 &= x^2 y^2 z^2, \\ \Phi_9 &= xyz(x^2 - y^2)(y^2 - z^2)(z^2 - x^2).\end{aligned}$$

Table 4. Octahedral case.

¹¹ For instance, for the dihedral, octahedral and icosahedral groups replace s by s_0 or y by $-y$ for passage through the plane $y=0$. For the tetrahedral case replace xyz by yzx for passage through the plane $y-x=0$.

¹² To save space the somewhat lengthy catalogue of substitutions is suppressed here. For details see Klein¹, p. 43.

φ behave as even functions, while the Φ behave as odd functions. The invariant Φ is simply the product of the totality of symmetry planes of each polyhedron. Any product of a Φ with some power or product of φ 's (of course of the same group) will also change sign at a symmetry plane. We shall call all (primitive or imprimitive) invariants made up of φ 's only *even* invariants, while invariants containing Φ in the *first* power shall be called odd invariants. The square of Φ is clearly an even invariant and as such must be connected by an identity¹³ with powers and products of the primitive even φ . The explicit derivation presents no difficulty except perhaps for the icosahedron for which case it is to be found on p. 219 of Kleins book.

Besides multiplication and addition one may employ invariant operators in order to form new invariants by differentiation. It is evident, due to the fact that the gradient transforms like a vector, that the operators obtained when replacing x by $\partial/\partial x$ etc. are invariant operators. We shall denote an operator, which has been obtained by this replacement, from an invariant $\varphi_l(xyz)$ or $\Phi(xyz)$ by $\bar{\varphi}_l$ or $\bar{\Phi}_l$. The degree of an invariant is depressed by l upon the application of such an operator. Most important for the present purpose is the operator $\bar{\varphi}_2$, which is the Laplacian. As for oddness and evenness the following rules are evident: The operator formed by replacing x by $\partial/\partial x$ etc. in an even φ is even, the operator formed from an odd Φ is odd. An even operator acting on an even or odd invariant will produce an even or odd invariant respectively. Conversely an odd operator acting on a even or odd invariant will produce an odd or even invariant.

The following two identities are the result of the previous statements

$$\bar{\varphi}_l \frac{1}{r} = r^{-2l-1} \varphi'_l \quad (1)$$

and

$$\bar{\Phi}_l \frac{1}{r} = r^{-2l-1} \Phi'_l, \quad (1')$$

where φ'_l and Φ'_l are otherwise arbitrary, even or odd operators of degree l . If an operator of degree l acts on $1/r$ an invariant of degree $-l-1$

¹³ Of degree $2n+2$, 12, 18, 30 respectively for the cases of tables 2 to 5.

results. But after differentiations the denominator r^{2l+1} has been built up, so the numerator must be an invariant of degree l .

3. Methods for obtaining polyhedral harmonics

We can now specify the desired properties of polyhedral harmonics. We wish to construct homogeneous polynomial solutions of Laplace's equation made up of invariants of each polyhedral case. These solid harmonics will form two sets according to whether they vanish on the symmetry planes (odd harmonics), or whether their normal derivative vanishes on these planes (even harmonics). It is clear that the odd harmonics should contain the first power of the primitive odd operator Φ of each group as a factor, while the even ones should be free of Φ . The lowest odd harmonic is the primitive Φ invariant itself, for upon operating on it with the Laplacian $\bar{\varphi}_2$ an odd operator of degree two less should result, which does not exist. The lowest even harmonic is the invariant $\varphi_0 = 1$.

For obtaining the higher harmonics two methods present themselves, which will be demonstrated on the octahedral case. For the other polyhedral merely the results will be given.

1. Odd harmonics

The lowest harmonic of this kind is Φ_9 ; all higher harmonics have to contain the first power of Φ_9 as factor. The first method begins with the construction of all odd invariants regardless of whether they are solutions of the Laplace equation. The following table is obtained:

degree	invariant	N_l
9	Φ_9	1
11	$\Phi_9 \Phi_2$	0
13	$\Phi_9 \varphi_2^2, \Phi_9 \varphi_4$	1
15	$\Phi_9 \varphi_2^3, \Phi_9 \varphi_1 \varphi_2, \Phi_9 \varphi_6$	1
17	$\Phi_9 \varphi_2^4, \Phi_9 \varphi_1 \varphi_2^2, \Phi_9 \varphi_4^2, \Phi_9 \varphi_6 \varphi_2$	1

Table 6.

The number of invariants is evidently equal to the number of partitions of $l-9$ into 2's, 4's and 6's. Now according to section 2, the Laplace operator applied to one of the above invariants

must give a linear combination of the above invariants but of degree two less. For instance

$$\bar{\varphi}_2 \bar{\Phi}_9 \varphi_2^2 = C_1 \bar{\Phi}_9 \varphi_2$$

and

$$\bar{\varphi}_2 \bar{\Phi}_9 \varphi_4 = C_2 \bar{\Phi}_9 \varphi_2;$$

therefore a certain linear combination of $\bar{\Phi}_9 \varphi_2^2$ and $\bar{\Phi}_9 \varphi_4$ will be a harmonic. Thus it is seen that the number of harmonics N_l is equal to the number of odd invariants of degree l minus that of degree $l-2$. This N_l is recorded in the last column of Table 6.

The second method adapts Maxwells famous method of poles to the present purpose. Since the Laplacian $\bar{\varphi}_2$ and any other operator $\bar{\varphi}$ or $\bar{\Phi}$ commute with one another, the identity (1') may be used substituting for $\bar{\Phi}_l$ the $\bar{\Phi}_9$ of the octahedron. Since no other odd invariant function besides $\bar{\Phi}_9$ exists, (1') becomes in this case

$$\bar{\Phi}_9 \frac{1}{r} = \text{const} \frac{1}{r^{19}} \bar{\Phi}_9.$$

Operating on this relation with any other even invariant operator, will by the same reasoning return numerator functions, which must contain $\bar{\Phi}_9$ as factor and which will therefore be odd harmonics. The operator $\bar{\varphi}_2$ constitutes of course an exception to this, since

$$\bar{\varphi}_2 \bar{\Phi}_9 \frac{1}{r} = \bar{\Phi}_9 \bar{\varphi}_2 \frac{1}{r} = 0.$$

Therefore only the operators of the following kind may be employed: sums of powers and products of $\bar{\varphi}_4$ and $\bar{\varphi}_6$ multiplied by $\bar{\Phi}_9$. In this way table 7 is obtained.

The last column indicates the number of operators giving a harmonic of degree l . It agrees with that of table 6. As a function of l , N_l is equal to the number of partitions of $l-9$ into 4's and 6's. The second method gives the harmonics themselves, but for practical computation the first method is perhaps to be preferred.

2. Even harmonics

As before we begin with a table of invariants, but now $\bar{\Phi}_9$ is to be omitted so as to get only even functions.

degree l	Harmonic	N_l
9	$\bar{\Phi}_9 \frac{1}{r}$	1
11	—	0
13	$\bar{\varphi}_4 \bar{\Phi}_9 \frac{1}{r}$	1
15	$\bar{\varphi}_6 \bar{\Phi}_9 \frac{1}{r}$	1
17	$\bar{\varphi}_4^2 \bar{\Phi}_9 \frac{1}{r}$	1
19	$\bar{\varphi}_4 \bar{\varphi}_6 \bar{\Phi}_9 \frac{1}{r}$	1
21	$\bar{\varphi}_4^3 \bar{\Phi}_9 \frac{1}{r}; \bar{\varphi}_6^2 \bar{\Phi}_9 \frac{1}{r}$	2

Table 7.

degree	invariant	N_l
0	1	1
2	φ_2^2	0
4	φ_2^2, φ_4	1
6	$\varphi_2^3, \varphi_4 \varphi_2, \varphi_6$	1

Table 8.

degree l	Harmonic	N_l
0	$\bar{1} \frac{1}{r}$	1
2	—	0
4	$\bar{\varphi}_4 \frac{1}{r}$	1
6	$\bar{\varphi}_6 \frac{1}{r}$	1

Table 9.

By the same reasoning as before it is seen that there exists a certain linear combination of φ_2^2 and φ_4 , whose Laplacian vanishes. So the first non-trivial even harmonic is of degree 4. The table is quite like table 6 after omitting $\bar{\Phi}_9$ in the latter. Table 9 uses the operator method and is very similar to table 7.

Evidently the number of even harmonics of degree l equals the number of odd harmonics of degree $l+9$.

4. Tables of Harmonics

The methods of the preceding section make it possible to calculate both even and odd polyhedral harmonics for all of the four groups. The notation used is that of tables 3 to 5 with the exception of the dihedral case, where the notation of surface harmonics is employed.

Dihedral case, $n \geq 2$

1. Odd harmonics:

$$P_l^{mn}(\cos \Theta) \sin m n \varphi,$$

$$l + m n = \text{odd integer},$$

$$m \geq 1.$$

The lowest harmonic both as to l and m is

$$\Phi_{n+1} = r^{n+1} P_{n+1}^n(\cos \Theta) \sin n \varphi.$$

2. Even harmonics:

$$P_l^{mn}(\cos \Theta) \cos m n \varphi,$$

$$l + m n = \text{even integer},$$

$$m \geq 0.$$

The lowest harmonic is $P_0 = 1$.

The numbers N_l of even and odd harmonics of degree l are connected through

$$N_{l+n+1}^{\text{odd}} = N_l^{\text{even}};$$

the latter is equal to the number of partitions of l into 2's and n 's, i.e. equal to the coefficient of x^l in the development

$$\frac{1}{1-x^2} \frac{1}{1-x^n} = \sum_{l=0}^{\infty} N_l^{\text{even}} x^l.$$

Octahedral case

This case is somewhat simpler than the tetrahedral case which will be discussed later.

1. Odd harmonics:

degree l	Harmonic
9	Φ_9
11	—
13	$(23 \varphi_4 - 5 \varphi_2^3) \Phi_9$
15	$(250 \varphi_6 - 75 \varphi_4 \varphi_2 + 29 \varphi_2^3) \Phi_9$

Table 10.

This table gives the result of the calculations which are implied in the operations of tables 6 and 7.

2. Even harmonics:

degree l	Harmonic
0	$\varphi_0 = 1$
2	—
4	$5 \varphi_4 - \varphi_2^2$
6	$231 \varphi_6 - 21 \varphi_4 \varphi_2 + 2 \varphi_2^3$

Table 11.

As for the number of harmonics for a given l we have the relation

$$N_{l+9}^{\text{odd}} = N_l^{\text{even}},$$

where the latter is given by

$$\frac{1}{(1-x^4)(1-x^6)} = \sum_{l=0}^{\infty} N_l^{\text{even}} x^l.$$

The first degeneracy, i.e. an $N_l > 1$, occurs in the set of table 11 for $l=12$ and in that of table 10 for $l=21$.

Tetrahedral case

Since the elementary spherical triangle of this case is much larger than that of the octahedral case (see table 1), the harmonics will be more densely spaced. Also since this group is a subgroup of the octahedral group, it is to be expected that the present case will have harmonics in common with the previous one. Inspection of table 3 shows that *two sets* of odd harmonics (i.e. of harmonics vanishing on the six tetrahedral planes) exist: one set of even degree possessing Φ_6 and another of odd degree possessing $\Phi_6 \varphi_3$ as factor. But since

$$\Phi_6^{\text{Tetr.}} \varphi_3^{\text{Tetr.}} = \Phi_9^{\text{Oct.}},$$

it is seen that this latter set is the set of octahedral (odd) harmonics of table 10. Similarly even harmonics of even degrees starting with φ_0 and of odd degrees starting with φ_3 are possible, the first set having already been listed in table 11. When regarding the harmonics of tables 10 and 11 as tetrahedral harmonics, one should, to be

consistent, change the symbol φ_6 wherever it occurs, to φ_3^2 and Φ_9 to $\varphi_3\Phi_6$.

1. Odd harmonics:

a. Even l values.

degree l	Harmonic
6	Φ_6
8	—
10	$(17\varphi_4 - 3\varphi_2^2)\Phi_6$
12	$(437\varphi_3^2 - 19\varphi_4\varphi_2 + 2\varphi_2^3)\Phi_6$
14	$(45\varphi_4^2 - 14\varphi_4\varphi_2 - 20\varphi_3^2\varphi_2 + \varphi_2^4)\Phi_6$

Table 12.

The first degeneracy occurs for $l=18$.

b. Odd l values. See table 10.

2. Even harmonics:

a. Odd l values.

degree l	Harmonic
3	φ_3
5	—
7	$(11\varphi_4 - 3\varphi_2^2)\varphi_3$
9	$(34\varphi_3^2 - 6\varphi_4\varphi_2 + \varphi_2^3)\varphi_3$

Table 13.

The first degeneracy occurs for $l=15$.

b. Even l values. See table 11.

The number of tetrahedral harmonics obeys the relation

$$N_{l+6}^{\text{odd}} = N_l^{\text{even}},$$

the latter being given as partition number into 3's and 4's by the development:

$$\frac{1}{(1-x^3)(1-x^4)} = \sum_{l=0}^{\infty} N_l^{\text{even}} x^l.$$

The first degeneracy occurs therefore for $l=12$.

All of the functions of tables 10 and 13 are seen to vanish on the planes $x=0$, $y=0$, $z=0$. They are therefore some of the harmonics of odd type belonging to the dihedral case with $n=2$. The reason for this is that this group, which is

called the four-group, is a subgroup of the tetrahedral and of the octahedral group.

Icosahedral case

1. Odd harmonics:

degree l	Harmonic
15	Φ_{15}
17	—
19	—
21	$(37\varphi_6 - 5\varphi_2^3)\Phi_{15}$
23	—

Table 14.

2. Even harmonics:

degree l	Harmonic
0	$\varphi_0 = 1$
2	—
4	—
6	$21\varphi_6 - 5\varphi_2^3$
8	—
10	$153\varphi_{10} - 675\varphi_3\varphi_2^2 - 350\varphi_2^5$

Table 15.

Calling N_l as before the number of harmonics of degree l , we have

$$N_{l+15}^{\text{odd}} = N_l^{\text{even}},$$

which is equal to the number of partitions of l into 6's and 10's. We have therefore the generating function:

$$\frac{1}{(1-x^6)(1-x^{10})} = \sum_{l=0}^{\infty} N_l^{\text{even}} x^l.$$

From this formula the nonexistence of even harmonics of degree 2, 4, 8, 14 and of odd ones of degree 17, 19, 23, 29, is immediately seen. The first degeneracy occurs evidently for $l=30$, and in the odd case for $l=45$.

5. Nodal lines for a few octahedral harmonics

Any spherical harmonic divides the surface of the sphere into variously shaped regions, the

dividing lines being called nodal lines. When considering the full surface of the sphere, the Legendre polynomials multiplied by suitable functions of the longitude angle divide the sphere by means of meridians and parallels in a familiar manner¹⁴. It seems desirable to gain some similar information for our harmonics so as to make them more tangible. We have for this purpose chosen the octahedral case, although other cases might just as readily have been used. Fig. 1 shows the intersections of the symmetry planes with a sphere in stereographic projection, in which circles become circles or straight lines. The edges of the octahedron are drawn as heavy lines, one of its corners is in the center of the figure, while the opposite corner is transformed to infinity. The edges of the dual cube are drawn as light lines so that its six faces are clearly discernible provided the region outside the four large circles is counted as a face. Where the symmetry circles cease to be cube edges, they have been continued as dashed lines. A cube face is thus intersected by two (dashed) diagonals and two (heavy) circles connecting the centers of opposite sides and an octahedron face by three medians. The entire surface of the sphere is therefore divided into the 48 triangles (table 1), each being bounded by one heavy, one light and one dashed arc. The fact that they are congruent is of course lost in the projection.

We begin the discussion of the harmonics with the odd Φ_9 . When we change this into a surface harmonic by division with r^9 and plot the projection, we get a surface, which is positive throughout one triangle and negative in the adjacent ones and which vanishes on all nine circles. Turning to the next harmonic of the set, viz. $(23\varphi_4 - 5\varphi_2^2)\Phi_9$, we see that the factor of Φ_9 causes the appearance of three nodal cones, whose traces upon the sphere are marked by six¹⁵ dotted curved nodal lines. These latter curves intersect every symmetry circle at a right angle and divide the sphere surface into 96 regions, in which the harmonic is alternately positive and negative. When we come to the harmonic of 15th degree (see table 10), then instead of the dotted

nodal curves which the figure shows, there will appear eight closed curves (not shown in the figure) surrounding the cube corners.

It will be recalled that the even harmonics are symmetric with respect to the symmetry circles. The first harmonic with nodal lines will be $5\varphi_4 - \varphi_2^2$ of table 11. It vanishes on the sphere on six closed curves, which are very similar in shape to the dotted curves of fig. 1, but nowhere else. The next harmonic is of degree 6 and has as nodal lines twelve closed curves, which this time surround the midpoints of the cube's edges.

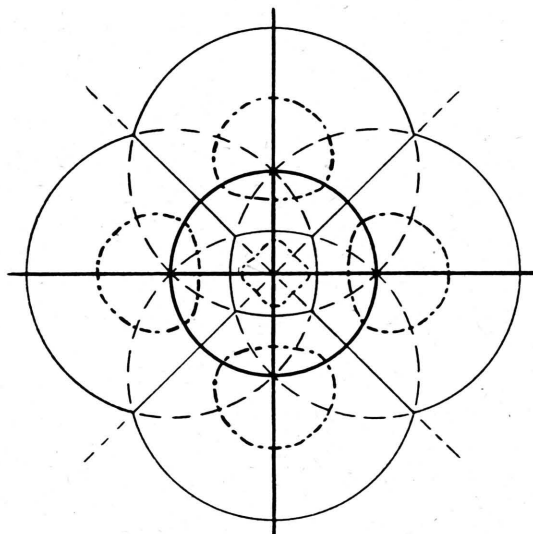


Fig. 1.

The behavior of the nodal lines seems from the few examples considered here rather unpredictable; it might be interesting to calculate a few higher cases and to find a general rule. The degenerate cases (like $l = 12$) should be even more interesting.

6. A theorem concerning the lowest odd harmonic

The lists of harmonics show that as the order l increases the sphere becomes subdivided into more and more regions. For the lowest odd harmonic it is possible to formulate this relation as a *theorem*. For the lowest odd harmonic Φ_l the order l is equal to the quotient of the perimeter p and the area A of the spherical triangle, which results from the division of the unit sphere by the symmetry planes of the polyhedral group.

¹⁴ See for instance HUND, in „Handbuch der Physik“, vol. 24—1, p. 578.

¹⁵ There should be a sixth closed curve, the opposite of the innermost one, surrounding the whole figure. Its inclusion in the figure would have necessitated reducing the scale considerably.

The proof consists merely in calculating the perimeter¹⁶ for the elementary triangles of table 1. Whether this theorem is generalizable, is a question, about which one can only speculate. It is certainly not true for the circular cones, into which the Legendre polynomials $P_l(\cos \Theta)$ divide the sphere. So perhaps p/A will have l as upper bound. But it remains to be investigated, what relation p/A has to the l of a harmonic belonging to polygons of great circles other than those of table 1.

As a conclusion we should like to point out a few unsolved problems connected with polyhedral harmonics. The classical researches by Schwarz¹⁷ and others¹⁸ on the connections between the polyhedra and the hypergeometric function naturally suggest the search for further and related classes of spherical harmonics. We are referring now to such harmonics as would belong to spherical triangles or polygons arising when one places one or several of the triangles of table 1 side by side. Of course, any of the harmonics of section 5 will also be a harmonic of such a wider region, but we are now asking for harmonics that are not contained in the previous sets, especially for the "fundamental", which does not vanish anywhere in the interior. Now it is easy to see (for the octahedral case with fig. 1) that whenever we make up a larger spherical figure by putting several of our elementary triangles side by side, we shall either get a triangle whose harmonics we already know, or we shall have a spherical polygon, which by repeated reflection at its boundaries will cover the sphere more than once. This latter fact becomes clear when one realizes that in such cases an odd number of regions will meet at certain corners, these being the corners where the primitive triangles have the angles $\pi/3$ and $\pi/5$.

¹⁶ This can be done best from the angles α, β, γ as given in table 1, by means of the formula

$$2 \sin \frac{p}{2} = \frac{[-\cos S \cos(S-\alpha) \cos(S-\beta) \cos(S-\gamma)]^{1/2}}{\sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}},$$

where $2S = \alpha + \beta + \gamma$.

¹⁷ H. A. Schwarz, *Gesammelte Abhandlungen*, vol. 2, p. 211.

¹⁸ See the presentation by F. Klein, *Über die hypergeometrische Funktion*, Berlin 1933, p. 257.

Therefore the harmonic will have changed sign after once having circumvented such a corner, so that a multiple round trip is necessary to establish continuity. The harmonic for such a region should therefore possess a number of points of ramification. Solutions of the three dimensional Laplace equation having points of ramification and which therefore cover the sphere more than once are up to the present known only for the trivial case where these points are opposite one another. Such a harmonic is for instance $P_{3/2}^{3/2}(\cos \Theta) \sin(3\varphi/2)$, the triangle of which has angles $(2/3, 2/3, 1)\pi$ and covers by reflection the sphere twice. Or the harmonic $P_{5/3}^{2/3}(\cos \Theta) \sin(2\varphi/3)$ belonging to the triangle $(3/4, 1/2, 1/2)\pi$, which after eight reflections covers the sphere three times. For all of these harmonics the lower index l is, in agreement with the above theorem, equal to the quotient of perimeter and area.

But more than two points of ramification will be necessary for harmonics, which belong to triangles made up of two or more adjacent elementary triangles, such as the triangles $(1/3, 1/3, 2/3)\pi$, $(1/4, 1/4, 2/3)\pi$ and $(1/3, 1/3, 2/5)\pi$. Here it should also be mentioned that one does not know the harmonics which belong to the whole tetrahedral face $(2/3, 2/3, 2/3)\pi$, the whole cube face $(2/3, 2/3, 2/3, 2/3)\pi$, the whole dodecahedral face $(2/3, 2/3, 2/3, 2/3, 2/3)\pi$, and the whole icosahedral face $(2/5, 2/5, 2/5)\pi$. The harmonics for these cases would, because of their high symmetry, be of special interest.

Up to here, all conal regions discussed, even though they may fill the space more than once, are concave. But there should be comparatively simple harmonics for cones whose opening solid angles are larger than 2π steradians. We are thinking here of solid angular regions whose spherical triangle has the area 4π minus the area of the triangles of table 1. With the help of such functions it should then be possible to construct characteristic functions for the spaces outside these polyhedral cones. If one assumes the validity of the theorem of this section, the degrees of all such ramified harmonics will be fractional. It is intended, in the near future, to determine the degrees l of harmonics of this type by means of an approximative procedure.